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# Commutators with Coefficients in CMO of Weighted Hardy Operators in Generalized Local Morrey Spaces

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**Abstract.** We prove theorems on the boundedness of commutators  $[a, H_w^\alpha]$  of the weighted multidimensional Hardy operator  $H_w^\alpha := wH_w^{\alpha-\frac{1}{w}}$  from a generalized local Morrey space  $\mathcal{L}^{p,\varphi;0}(\mathbb{R}^n)$  to local or global space  $\mathcal{L}^{q,\psi}(\mathbb{R}^n)$ . The main impacts of these theorems are

1. the use of  $\text{CMO}_s$ -class of coefficients  $a$  for the commutators;
2. the general setting when the function  $\varphi$  defining the Morrey space and the weight  $w$  are independent of one another and the weight  $w$  is not assumed to be in  $A_p$ ;
3. recovering the Sobolev–Adams exponent  $q$  instead of Sobolev–Spanne type exponent in the case of classical Morrey spaces
4. boundedness from local to global Morrey spaces;
5. the obtained estimates contain the parameter  $s > 1$  which may be arbitrarily chosen. Its choice regulates in fact an equilibrium between assumptions on the coefficient  $a$  and the characteristics of the space.

The obtained results are new also in non-weighted case.

**Mathematics Subject Classification.** 46E30, 42B35, 42B25, 47B38.

**Keywords.** Morrey space, weighted Hardy operator, commutators, BMO, local BMO, weighted Hardy inequalities, quasi-monotone weights.

## 1. Introduction

The main object in this paper is the multi-dimensional Hardy operator

$$H_w^\alpha f(x) = |x|^{\alpha-n} w(|x|) \int_{|y|<|x|} \frac{f(y)dy}{w(|y|)}, \quad (1.1)$$

where  $w$  is a weight, and its commutators

$$[a, H_w^\alpha]f(x) := |x|^{\alpha-n} w(|x|) \int_{|y|<|x|} \frac{a(x) - a(y)}{w(|y|)} f(y)dy. \quad (1.2)$$

with coefficients  $a \in \text{CMO}_s$ , the class of central mean oscillation.

We obtain conditions for the boundedness of the weighted commutators (1.2) from a generalized local Morrey space to a generalized global Morrey space, with coefficients in the class  $\text{CMO}_s$ .

The spaces  $\mathcal{L}^{p,\lambda}$  which bear the name of *Morrey spaces* originate from the paper [19] by C. Morrey on regularity problems of solutions to PDE. A formulation of Morrey's ideas in terms of normed function spaces probably first appeared in [6]. Classical Morrey spaces are well presented in various books, see e.g. [10, 16] and references therein; we refer also to the recent survey paper [22].

There are also known the so-called *generalized Morrey spaces*  $\mathcal{L}^{p,\varphi}(\Omega)$ , defined by the norm

$$\|f\|_{p,\varphi} := \sup_{x \in \Omega, r > 0} \left( \frac{1}{\varphi(r)} \int_{\tilde{B}(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}}, \quad (1.3)$$

where  $\Omega \subseteq \mathbb{R}^n$  is an open set,  $\tilde{B}(x, r) = B(x, r) \cap \Omega$  and  $\varphi(r)$  is a non-negative measurable function on  $[0, \ell]$ ,  $\ell = \text{diam} \Omega$ , satisfying certain assumptions (one may take  $\varphi$  depending not only on  $r$ , but on the point  $x$  as well). We refer for instance, to [11] and also the survey [22] where one can find a detailed historical account on such spaces.

There is also known the version of Morrey spaces, called *local generalized Morrey spaces* denoted sometimes as  $\mathcal{L}_{x_0}^{p,\varphi}(\Omega)$  and defined by the norm

$$\|f\|_{p,\varphi;x_0} := \sup_{r > 0} \left( \frac{1}{\varphi(r)} \int_{\tilde{B}(x_0,r)} |f(y)|^p dy \right)^{\frac{1}{p}}, \quad x_0 \in \overline{\Omega}. \quad (1.4)$$

We refer, for instance, to [12, 25], but note that they appeared in [9] in the case of  $\varphi(r) = r^\lambda$ ,  $\Omega = \mathbb{R}^n$  and  $x_0 = 0$ .

Various versions of Morrey spaces were widely investigated during the past decades, including the study of classical operators of harmonic analysis - maximal, singular and potential operators in Morrey spaces and their generalizations and modifications, see for instance [3–5, 11, 15, 20, 21] and references therein.

In comparison, the spaces defined by the norm 1.3, are often called *global Morrey spaces*.

In the case  $\varphi(r) = r^\lambda$ ,  $\Omega = \mathbb{R}^n$  and  $x_0 = 0$  local Morrey spaces are also known ([1]) as *central Morrey spaces*.

With regard to the weighted operator (1.1), note that, given an operator  $A$  and a weight  $w$ , the boundedness of its weighted version, i.e.  $wA\frac{1}{w}$  in the Morrey space  $\mathcal{L}^{p,\varphi}$ , global or local, is equivalent to the boundedness of the operator  $A$  itself in the *weighted Morrey space*, defined, for instance in the global case, by the norm

$$\sup_{x \in \Omega, r > 0} \left( \frac{1}{\varphi(r)} \int_{\tilde{B}(x,r)} |w(y)f(y)|^p dy \right)^{\frac{1}{p}}, \quad (1.5)$$

i.e. in the form where the weight  $w$  and the function  $\varphi$  are independent of each other. We went into these details to avoid a misunderstanding in terminology: sometimes weighted Morrey spaces are introduced in a specific way, with the function  $\varphi$  depending on the weight  $w$ .

Estimates of the classical operators of harmonic analysis in weighted Morrey spaces in the natural setting, i.e. in the general form with the function  $\varphi$  and the weight  $w$  independent of each other, were less studied.

We find it important to note that the class of weights  $w$ , for which, for instance the maximal operator is bounded in the weighted Morrey space, depends, as observed in [23] (see also [24, 26]), on the function  $\varphi$ . In the case  $\varphi(0) = 0$  which differs Morrey spaces from Lebesgue spaces, the class of admissible weights is different from the Muckenhoupt class  $A_p$ . It is larger than  $A_p$  with respect to possible vanishing of weights, but more narrow with respect to growth of weights. Therefore, the assumption for the weight to belong to  $A_p$  made in various studies on Morrey spaces is not natural for the weighted setting of problems for Morrey spaces. We avoid such a restriction in our approach.

In our estimations of the weighted commutators (1.2) we base ourselves on the approach developed in [18, 21, 23, 25], to obtain pointwise estimates for the Hardy operator.

Note that the boundedness results for weighted commutators of multidimensional Hardy operators are known for Lebesgue spaces  $L^p(\mathbb{R}^n)$ , see [8]. For Morrey spaces norm estimates were obtained in non-weighted case ( $w \equiv 1$ ) in [7] for classical Morrey space, i.e. in the case where  $\varphi$  is a power function.

Our progress in proving the boundedness of weighted commutators of Hardy operators within the frameworks of generalized Morrey spaces is based on the fact that we are able to prove the direct pointwise estimate of the commutator  $[a, H_w^\alpha]f$  itself via the product of the norm  $\|f\|_{\mathcal{L}^{p,\varphi;0}}$  by a concrete function depending only on the coefficient  $a$ , the weight  $w$  and the parameters  $p, \varphi$  of the space, see Theorem 3.1. This enables us to immediately pass to Morrey norms of commutator in this estimate and thereby obtain the main result on the boundedness, where on the right-hand side of the estimate there appears the Morrey norm of the above concrete function, see Theorems 3.3 and 3.4.

The paper is organized as follows. In Sect. 2 we recall the notion of  $\text{CMO}_p$ -spaces (spaces of central mean oscillation) and prove important technical Lemma 2.1 for functions in  $\text{CMO}_p$ . We introduce also the class of weights which we use in this paper. In Sect. 3 we prove the main results (Theorems 3.1–3.4 and 3.6) on the boundedness of the commutators of weighted multidimensional Hardy operators, with coefficients in CMO-type spaces. Among these theorems, the main one in some sense is the final Theorem 3.6. In this

theorem we impose restrictions on the functions  $\varphi$  and the ratio  $\frac{\varphi^{\frac{1}{p}}}{w}$  which are just slightly more restrictive than in Theorem 3.4, but which allow to radically simplify the final assumptions on characteristics  $p$  and  $\varphi$  of the space and the weight  $w$ . The proof of the above mentioned pointwise estimate of commutators is given in Theorem 3.1. In Corollary 3.9 we provide the corresponding result for the case of the classical Morrey spaces, when  $\varphi(r) = r^\lambda$ .

Everywhere in the sequel we assume that the function  $\varphi$  defining Morrey space is continuous in a neighborhood of the origin, almost increasing and satisfies the following conditions:

$$\varphi(0) = 0 \text{ and } \inf_{\delta < r < \infty} \varphi(r) > 0 \quad \text{for every } \delta > 0. \quad (1.6)$$

If  $\varphi(0) > 0$  and  $\varphi(r)$  does not tend to 0 at infinity, the generalized Morrey space is nothing else but the Libesgue space  $L^p(\mathbb{R}^n)$ .

## 2. Preliminaries

### 2.1. On $\text{CMO}_p$ -Spaces

The  $\text{BMO}$ -space, as is well known, is defined by the quasi-norm

$$\|f\|^* = \sup_{x \in \mathbb{R}^n} \sup_{r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(z) - f_{B(x, r)}| dz, \quad (2.1)$$

where  $f_{B(x, r)} := \frac{1}{|B(x, r)|} \int_{B(x, r)} f(z) dz$ . The  $\text{BMO}$  is an appropriate class of coefficients for commutators of many classical operators, in particular, for integral operators whose kernels have singularity on the whole diagonal  $x = y$ , such as singular and potential operators.

For Hardy type operators with singularities of the kernel only at the origin  $x = y = 0$  and infinity  $x = y = \infty$ , a wider class of coefficients, with  $\text{BMO}$ -type-behaviour only at the origin, is more appropriate. Such a local version of  $\text{BMO}$ , the space  $\text{CMO}$  (*central mean oscillation*) is defined by the norm

$$\|f\|^{*,0} = \sup_{r > 0} \frac{1}{|B(0, r)|} \int_{B(0, r)} |f(z) - f_{B(0, r)}| dz, \quad (2.2)$$

We also need its generalization, the space  $\text{CMO}_p$  depending on  $p$ , defined by the norm

$$\|f\|_p^{*,0} = \sup_{r > 0} \left( \frac{1}{|B(0, r)|} \int_{B(0, r)} |f(z) - f_{B(0, r)}|^p dz \right)^{\frac{1}{p}}, \quad (2.3)$$

so that  $\text{CMO} = \text{CMO}_p|_{p=1}$ . However, in contrast to the “global”  $\text{BMO}$ -space, such local  $\text{CMO}_p$ -spaces no more are independent of  $p$ . By Jensen inequality we have

$$\|f\|^{*,0} \leq \|f\|_p^{*,0} \leq \|f\|_q^{*,0}$$

and  $\text{BMO} \subset \text{CMO}_q \subset \text{CMO}_p \subset \text{CMO}$ ,  $1 < p < q < \infty$ .

We refer to [1, 13, 17] for the study of the classes  $\text{CMO}_p$ .

The following property:

$$|a_{B(x,r)} - a_{B(x,t)}| \leq C \|a\|^{*,0} \ln \frac{t}{r} \quad \text{for } 0 < 2r < t, \quad (2.4)$$

of functions  $a$  in the global space  $BMO$  is known which goes back to [14]. This property proves to be also true in the local setting, as shown in the following lemma:

**Lemma 2.1.** *Let  $a \in CMO$ . Then*

$$|a_{B(0,r)} - a_{B(0,t)}| \leq C \|a\|^{*,0} \left(1 + \ln \frac{t}{r}\right) \quad \text{for } 0 < r \leq t, \quad (2.5)$$

where one can take  $C = 2(1 + 2^n)$ .

*Proof.* At the first step we obtain

$$|a_{B(0,r)} - a_{B(0,\lambda r)}| \leq (1 + \lambda^n) \|a\|^{*,0} \quad (2.6)$$

for all  $\lambda \geq 1$ . Indeed

$$\begin{aligned} a_{B(0,r)} - a_{B(0,\lambda r)} &= \frac{1}{B(0,r)} \int_{B(0,r)} (a(y) - a_{B(0,\lambda r)}) \, dy \\ &\quad - \frac{1}{B(0,\lambda r)} \int_{B(0,\lambda r)} (a(y) - a_{B(0,\lambda r)}) \, dy \end{aligned}$$

Hence

$$|a_{B(0,r)} - a_{B(0,\lambda r)}| \leq \frac{|B(0,\lambda r)|}{|B(0,r)|} \|a\|^{*,0} + \|a\|^{*,0}$$

which gives (2.6). Since  $|a_{B(0,r)} - a_{B(0,\lambda^m r)}| \leq \sum_{j=0}^{m-1} |a_{B(0,\lambda^j r)} - a_{B(0,\lambda^{j+1} r)}|$ , we then get

$$|a_{B(0,r)} - a_{B(0,\lambda^m r)}| \leq m(1 + \lambda^n) \|a\|^{*,0} \quad (2.7)$$

for all  $\lambda \geq 1$  and  $m = 1, 2, 3, \dots$

Let first  $t \geq 2$ . We represent  $t$  as  $t = 2^{N+\alpha}r$ , where  $N + \alpha = \log_2 \frac{t}{r}$  and  $N = \lfloor \log_2 \frac{t}{r} \rfloor$ . and proceed as follows:

$$|a_{B(0,r)} - a_{B(0,t)}| \leq |a_{B(0,r)} - a_{B(0,2^N r)}| + |a_{B(0,2^N r)} - a_{B(0,2^{N+\alpha} r)}|,$$

where the first term is estimated by (2.7) with  $\lambda = 2$  and  $m = N$ , and the second one by (2.7) with  $r$  replaced by  $r2^N$ ,  $\lambda = 2^\alpha \in [1, 2)$  and  $m = 1$  so that we arrive at (2.5).

In the remaining case  $r \leq t \leq 2r$ , it suffices to observe that  $|a_{B(0,r)} - a_{B(0,t)}| \leq C \|a\|^{*,0}$  by (2.6) with  $\lambda - \frac{t}{r} \in (1, 2)$ .  $\square$

## 2.2. On Quasi Monotone Functions

In the sequel, a non-negative function  $f$  on  $[0, \ell]$ ,  $0 < \ell \leq \infty$ , is called almost increasing (almost decreasing), if there exists a constant  $C(\geq 1)$  such that  $f(x) \leq Cf(y)$  for all  $x \leq y$  ( $x \geq y$ , respectively). Equivalently, a function  $f$  is almost increasing (almost decreasing), if it is equivalent to an increasing (decreasing, resp.) function  $g$ , i.e.  $c_1 f(x) \leq g(x) \leq c_2 f(x)$ ,  $c_1 > 0$ ,  $c_2 > 0$ .

- Definition 2.2.** 1. By  $W = W(\mathbb{R}_+)$  we denote the class of functions  $\varphi$  continuous and positive on  $\mathbb{R}_+$  such that there exists the finite limit  $\lim_{x \rightarrow 0} \varphi(x)$ ;
2. by  $W_0 = W_0(\mathbb{R}_+)$  we denote the class of functions  $\varphi \in W$  almost increasing on  $(\mathbb{R}_+)$ ;
3. by  $\overline{W} = \overline{W}(\mathbb{R}_+)$  we denote the class of functions  $\varphi \in W$  such that  $x^a \varphi(x) \in W_0$  for some  $a = a(\varphi) \in \mathbb{R}$ ;
4. by  $\underline{W} = \underline{W}(\mathbb{R}_+)$  we denote the class of functions  $\varphi \in W$  such that there exists a number  $b \in \mathbb{R}$  such that  $\frac{f(t)}{t^b}$  is almost decreasing.

By  $\Delta_2$  we denote the class of non-negative functions  $\varphi$  on  $\mathbb{R}_+$  satisfying the doubling condition  $\varphi(2t) \leq C\varphi(t)$ .

Note that  $\underline{W} \subset \Delta_2$ .

Everywhere in the sequel we assume that the function  $\varphi$  defining the Morrey space and the weight  $w$  satisfy the condition

$$\varphi, \frac{1}{w} \in \overline{W} \quad \text{and} \quad \frac{\varphi^{1/p}}{w} \in \mathcal{W}(\mathbb{R}_+), \quad (2.8)$$

which is equivalent to saying that the functions  $\varphi$  and  $w$  have finite Matuszewska–Orlicz indices. Note also that the assumption that  $\varphi$  is almost increasing yields that the indices of the function  $\varphi$  are non-negative.

Observe also that sums, products, quotients and powers of functions in  $\underline{W} \cap \overline{W}$  are also in  $\underline{W} \cap \overline{W}$ .

We will also use the notation

$$\mathcal{W} := (\overline{W} \cap \Delta_2) \cup \underline{W}.$$

Since  $\underline{W} = \underline{W} \cap \Delta_2$ , we have

$$\mathcal{W} = (\overline{W} \cup \underline{W}) \cap \Delta_2 \subset \Delta_2.$$

**Lemma 2.3.** *Let  $h \in L^1_{\text{loc}}(\mathbb{R}_+)$  be non negative. If  $h \in \Delta_2$ , then*

$$t^\gamma \int_0^t h(\tau) d\tau \in \Delta_2 \bigcap \overline{W}$$

for every  $\gamma \in \mathbb{R}$ .

*Proof.* Since  $\int_0^t h(\tau) d\tau$  is increasing, the inclusion  $t^a \int_0^t h(\tau) d\tau \in \overline{W}$  is obvious. It remains to note that  $h \in \Delta_2 \implies \int_0^t h(\tau) d\tau \in \Delta_2$ .  $\square$

**Lemma 2.4.** *Let  $g \in \mathcal{W}(\mathbb{R}_+)$ . Then there exists a constant  $C > 0$  such that*

$$\sum_{k=0}^{\infty} g(2^{-k}r) \leq C \int_0^r \frac{g(t)}{t} dt, \quad r \in \mathbb{R}_+ \quad (2.9)$$

under the assumption that the right-hand side integral exists.

*Proof.* This is in fact a known estimation of a series by an integral from mathematical analysis, usually proved for monotonic functions  $g$  (then it holds with  $C = 1$ ), but the proof is essentially the same, which we provide below for the completeness of presentation.

We have

$$\int_0^r \frac{g(t)}{t} dt = \sum_{k=0}^{\infty} \int_{2^{-k-1}r}^{2^{-k}r} \frac{g(t)}{t} dt.$$

By definition of the classes  $\overline{W}$  and  $\underline{W}$ , it can be easily seen that

$$\begin{cases} g(t) \geq Cg(2^{-k-1}r), & \text{if } g \in \overline{W} \\ g(t) \geq Cg(2^{-k}r), & \text{if } g \in \underline{W} \end{cases} \quad \text{on } (2^{-k-1}r, 2^{-k}r)$$

for some constant  $C$ . Then

$$\int_0^r \frac{g(t)}{t} dt \geq C \ln 2 \sum_{k=0}^{\infty} g(2^{-k}r)$$

for  $g \in \mathcal{W}$ . □

By  $\Phi^\gamma(\mathbb{R}_+)$  we denote the class of functions  $\varphi \in \overline{W}(\mathbb{R}_+)$ ,  $\gamma \in \mathbb{R}$  such that  $\int_0^r \frac{f(t)}{1+t^\gamma} \leq C \frac{f(r)}{r^\gamma}$ ,  $0 < r < \infty$ . This class is known in the literature as Bari–Stechkin class [2]. Various properties of functions in this class may be found, for instance in [25], see also references therein. We recall only properties we use in this paper. To this end, we first introduce the indices

$$m(\varphi) = \sup_{0 < r < 1} \frac{\ln \left( \limsup_{h \rightarrow 0} \frac{\varphi(hr)}{\varphi(h)} \right)}{\ln r} = \lim_{r \rightarrow 0} \frac{\ln \left( \limsup_{h \rightarrow 0} \frac{\varphi(hr)}{\varphi(h)} \right)}{\ln r} \quad (2.10)$$

$$M(\varphi) = \sup_{r > 1} \frac{\ln \left( \limsup_{h \rightarrow 0} \frac{\varphi(hr)}{\varphi(h)} \right)}{\ln r} = \lim_{r \rightarrow \infty} \frac{\ln \left( \limsup_{h \rightarrow 0} \frac{\varphi(hr)}{\varphi(h)} \right)}{\ln r} \quad (2.11)$$

and

$$m_\infty(\varphi) = \sup_{r > 1} \frac{\ln \left[ \liminf_{h \rightarrow \infty} \frac{\varphi(rh)}{\varphi(h)} \right]}{\ln r}, \quad M_\infty(\varphi) = \inf_{r > 1} \frac{\ln \left[ \limsup_{h \rightarrow \infty} \frac{\varphi(rh)}{\varphi(h)} \right]}{\ln r} \quad (2.12)$$

for functions in  $W(\mathbb{R}_+)$ , known as Matuszevska-Orlicz indices.

The indices  $m(\varphi)$  and  $m_\infty(\varphi)$  are finite numbers when  $\varphi \in \overline{W}(\mathbb{R}_+)$  and  $M(\varphi)$  and  $M_\infty(\varphi)$  are finite numbers when  $\varphi \in \underline{W}(\mathbb{R}_+)$ . Besides this,

$$m(\varphi) = \sup \left\{ a : \frac{\varphi(t)}{t^a} \text{ is almost increasing on } (0, 1] \right\}, \quad (2.13)$$

and

$$m_\infty(\varphi) = \sup \left\{ a : \frac{\varphi(t)}{t^a} \text{ is almost increasing on } [1, \infty) \right\}, \quad (2.14)$$

$$M(\varphi) = \inf \left\{ a : \frac{\varphi(t)}{t^a} \text{ is almost decreasing on } (0, 1] \right\}, \quad (2.15)$$

and

$$M_\infty(\varphi) = \inf \left\{ a : \frac{\varphi(t)}{t^a} \text{ is almost decreasing on } [1, \infty) \right\}. \quad (2.16)$$

It is also known for  $\varphi \in \overline{W}$  that

$$\int_0^r \frac{\varphi(t)}{t^{1+\gamma}} dt \leq c \frac{\varphi(r)}{r^\gamma} \text{ on } (0, 1], \iff m(\varphi) > \gamma$$

and

$$\int_0^r \frac{\varphi(t)}{t^{1+\gamma}} dt \leq c \frac{\varphi(r)}{r^\gamma} \text{ on } [1, \infty), \iff m_\infty(\varphi) > \gamma$$

so that for  $\varphi \in \overline{W}(\mathbb{R}_+)$  we have

$$\varphi \in \Phi^\gamma \iff \min\{m(\varphi), m_\infty(\varphi) > \gamma\} \quad (2.17)$$

By (2.17) the property

$$\varphi \in \Phi^\gamma \iff \varphi^t \in \Phi^{t\gamma}, \quad t > 0 \quad (2.18)$$

holds. It is also useful to keep in mind that any function  $\varphi \in \overline{W}(\mathbb{R}_+)$  belongs to some class  $\Phi^\gamma(\mathbb{R}_+)$  (with  $\gamma$  depending on  $\varphi$ ). Namely,

$$\varphi \in \overline{W}(\mathbb{R}_+) \implies \varphi \in \Phi^\gamma(\mathbb{R}_+) \quad \text{for all } \gamma < \min\{m(\varphi), m_\infty(\varphi)\}. \quad (2.19)$$

### 3. Weighted Estimates of Commutators of Hardy Operators in Morrey Spaces

#### 3.1. Pointwise Estimates

In the following theorem we denote

$$\Phi(r) := \frac{r^{\frac{n}{p'}} \varphi^{\frac{1}{p}}(r)}{w(r)}$$

and

$$B_k(|y|) := \{z : |z| < 2^{-k}|y|\}.$$

**Theorem 3.1.** (Pointwise estimate) *Let  $1 \leq p < \infty$ , the functions  $\varphi$  and  $w$  satisfy the assumptions (1.6) and (2.8) and  $a \in CMO_{p'}$ . Then for all  $f \in \mathcal{L}^{p,\varphi}(\mathbb{R}^n)$  and  $y \in \mathbb{R}^n$  the following pointwise estimate holds*

$$\int_{|z| < |y|} \frac{|a(z) - a(y)|}{w(|z|)} |f(z)| dz \leq C \left( \|a\|_{p',0}^{*,0} \mathcal{A}(|y|) + \mathcal{B}_a(y) \right) \|f\|_{\mathcal{L}^{p,\varphi;0}} \quad (3.1)$$

where

$$\begin{aligned} \mathcal{A}(|y|) &:= \int_0^{|y|} \frac{\Phi(t)}{t} dt = \int_0^{|y|} \frac{t^{\frac{n}{p'}-1} \varphi^{\frac{1}{p}}(t)}{w(t)} dt, \\ \mathcal{B}_a(y) &:= \sum_{k=0}^{\infty} \Phi(2^{-k}|y|) |a(y) - a_{B_k(|y|)}| \end{aligned}$$



and  $C > 0$  does not depend on  $f, a, y$  and  $r$ .

*Proof.* Let  $R_k(|y|) = B_k(|y|) \setminus B_{k+1}(|y|) = \{z : 2^{-k-1}|y| < |z| < 2^{-k}|y|\}$ . We have

$$\begin{aligned} \int_{|z| < |y|} |a(z) - a(y)| \frac{|f(z)|}{w(|z|)} dz &= \sum_{k=0}^{\infty} \int_{R_k(|y|)} |a(z) - a(y)| \frac{|f(z)|}{w(|z|)} dz \\ &\leq \sum_{k=0}^{\infty} \int_{R_k(|y|)} |a(z) - a_{B_k(|y|)}| \frac{|f(z)|}{w(|z|)} dz \\ &\quad + \sum_{k=0}^{\infty} |a(y) - a_{B_k(|y|)}| \int_{R_k(|y|)} \frac{|f(z)|}{w(|z|)} dz. \end{aligned}$$

It is easily checked that  $w \in \underline{W} \cap \overline{W}$  implies  $\frac{1}{w(|z|)} \leq \frac{C}{w(2^{-k}|y|)}$  for  $z \in R_k(|y|)$ . Then

$$\begin{aligned} \int_{|z| < |y|} |a(z) - a(y)| \frac{|f(z)|}{w(|z|)} dz &\leq C \sum_{k=0}^{\infty} \frac{1}{w(2^{-k}|y|)} \int_{B_k(|y|)} |a(z) - a_{B_k(|y|)}| |f(z)| dz \\ &\quad + C \sum_{k=0}^{\infty} \frac{|a(y) - a_{B_k(|y|)}|}{w(2^{-k}|y|)} \int_{B_k(|y|)} |f(z)| dz, \end{aligned}$$

whence by Hölder inequality we get

$$\begin{aligned} &\int_{|z| < |y|} |a(z) - a(y)| \frac{|f(z)|}{w(|z|)} dz \\ &\leq C \sum_{k=0}^{\infty} \frac{1}{w(2^{-k-1}|y|)} \left( \int_{B_k(|y|)} |a(z) - a_{B_k(|y|)}|^{p'} dz \right)^{\frac{1}{p'}} \left( \int_{B_k(|y|)} |f(z)|^p dz \right)^{\frac{1}{p}} \\ &\quad + C \sum_{k=0}^{\infty} \frac{|a(y) - a_{B_k(|y|)}|}{w(2^{-k-1}|y|)} (2^{-k}|y|)^{\frac{n}{p'}} \left( \int_{B_k(|y|)} |f(z)|^p dz \right)^{\frac{1}{p}} : I_1 + I_2. \end{aligned}$$

By the definition of the norm in the Morrey space, we then obtain

$$I_1 \leq C \|a\|_{p',0}^{*,0} \|f\|_{\mathcal{L}^{p,\varphi;0}} \sum_{k=0}^{\infty} \frac{(2^{-k}|y|)^{\frac{n}{p'}} \varphi^{\frac{1}{p}}(2^{-k}|y|)}{w(2^{-k-1}|y|)} = C \|a\|_{p',0}^{*,0} \|f\|_{\mathcal{L}^{p,\varphi;0}} \sum_{k=0}^{\infty} \Phi(2^{-k}|y|).$$

By Lemma 2.4 with  $g(t) = \Phi(t)$  we obtain

$$I_1 \leq C \|a\|_{p',0}^{*,0} \|f\|_{\mathcal{L}^{p,\varphi;0}} \int_0^r \frac{\Phi(t)}{t} dt.$$

For  $I_2$  we have

$$I_2 \leq C \|f\|_{\mathcal{L}^{p,\varphi;0}} \sum_{k=0}^{\infty} \Phi(2^{-k}|y|) |a(y) - a_{B_k(|y|)}|.$$

It remains to gather the estimates for  $I_1$  and  $I_2$ . □

### 3.2. Main Results

From the pointwise estimate of Theorem 3.1 we immediately obtain the following statement on the boundedness of the weighted commutator of the Hardy operator from a local Morrey space  $L^{p,\varphi;0}(\mathbb{R}^n)$  to a global Morrey space  $L^{q,\psi}(\mathbb{R}^n)$ , where  $1 < q < \infty$  and  $\psi$  satisfies (1.6).

**Theorem 3.2.** *Under the assumptions of Theorem 3.1 the weighted commutator  $[a, H_w^\alpha]$  is bounded from the local space  $L^{p,\varphi;0}(\mathbb{R}^n)$  to the global space  $L^{q,\psi}(\mathbb{R}^n)$  if*

$$\mathcal{A}(|y|), \mathcal{B}_a(y) \in L^{q,\psi}(\mathbb{R}^n) \quad (3.2)$$

The condition (3.2) may be reduced to simple easily verified conditions, if we consider the action of the commutator not to a global Morrey space but to a local one, as given in Theorem 3.4 after the preparatory Theorem 3.3.

In the next theorem we use the following functions, defined by the function  $\varphi(r)$  and the weight  $w(r)$ :

$$\mathbb{A}(r) := \|t^{\alpha-n}w(t)\mathcal{A}(t)\|_{L^q(B(0,r))} = \left( \int_0^r \left| t^{\alpha-n}w(t) \int_0^t \frac{\tau^{\frac{n}{p'}-1}\varphi^{\frac{1}{p}}(\tau)}{w(\tau)} d\tau \right|^q t^{n-1} dt \right)^{\frac{1}{q}}, \quad (3.3)$$

$$\mathbb{B}_{s,\beta}(r) = r^{\alpha+\beta-\frac{n}{q'}} \int_0^r t^{\frac{n}{(qs')}-\alpha-\beta} \left( \int_0^t \tau^{(\alpha-\frac{n}{p'})qs'+n-1} \varphi^{\frac{qs'}{p}} d\tau \right)^{\frac{1}{qs'}} \frac{dt}{t}, \quad (3.4)$$

$$\mathbb{C}_\beta(r) = r^{\alpha+\beta-\frac{n}{q'}} \int_0^r t^{\frac{n}{q'}-\alpha-\beta} \left( \int_0^t \left| \tau^{\alpha-\frac{n}{p'}} \varphi^{\frac{1}{p}}(\tau) \left( 1 + \ln \frac{r}{\tau} \right) \right|^q \tau^{n-1} d\tau \right)^{\frac{1}{q}} \frac{dt}{t}, \quad (3.5)$$

where  $1 < s < \infty$ ,  $\beta \in \mathbb{R}$ .

In Theorems 3.3, 3.4 and 3.6 there is used the parameter  $s > 1$  which may be arbitrary chosen. Its choice regulates in fact an equilibrium between assumptions on the coefficient  $a$  and the characteristics of the space. The more  $s$  is, the more restriction we impose on  $a$  and less on the space, and viceversa.

**Theorem 3.3.** *Let  $1 < p < \infty$ ,  $1 < q < \infty$ ,  $a \in CMO_{p'} \cap CMO_{sq}$  for some  $s > 1$ , the conditions (1.6) and (2.8) be satisfied. Then*

$$\|[a, H_w^\alpha]f\|_{L^q(B(0,r))} \leq C \left( \|a\|_{p'}^{*,0} \mathbb{A}(r) + \|a\|_{qs}^{*,0} \mathbb{B}_{s,\beta}(r) + \|a\|^{*,0} \mathbb{C}_\beta(r) \right) \|f\|_{L^{p,\varphi;0}} \quad (3.6)$$

where  $\beta = \beta(w)$  is any number such that

$$\beta > \max\{M(w), M_\infty(w)\}. \quad (3.7)$$

*Proof.* We have

$$|[a, H_w^\alpha]f(y)| \leq |y|^{\alpha-n}w(|y|) \int_{|z|<|y|} \frac{|a(y) - a(z)|}{w(|z|)} |f(z)| dz,$$

and by the estimate (3.1) we get

$$|[a, H_w^\alpha]f(y)| \leq C|y|^{\alpha-n}w(|y|) \left[ \|a\|_{p'}^{*,0} \mathcal{A}(|y|) + \mathcal{B}_a(y) \right] \|f\|_{L^{p,\varphi;0}}. \quad (3.8)$$

Denote

$$\mathcal{C}(y) := |y|^{\alpha-n} w(|y|) \mathcal{A}(|y|) \quad \text{and} \quad \mathcal{D}_a(y) := |y|^{\alpha-n} w(|y|) \mathcal{B}_a(y)$$

for brevity, so that

$$\|[a, H_w^\alpha]\|_{L^q(B(0,r))} \leq C \left( \|a\|_{p'}^{*,0} \|\mathcal{C}\chi_{B(0,r)}\|_{L^q} + \|\mathcal{D}_a(y)\chi_{B(0,r)}\|_{L^q} \right) \|f\|_{\mathcal{L}^{p,\varphi;0}}$$

and we have to estimate the norms  $\|\mathcal{C}\chi_{B(0,r)}\|_{L^q}$  and  $\|\mathcal{D}_a\chi_{B(0,r)}\|_{L^q}$ . For the term with  $\mathcal{C}$  there is nothing more to estimate, it just generates the term  $\mathbb{A}(r)$  in (3.6). For another term we have

$$\mathcal{D}_a(y) \leq \mathcal{E}_a(y) + \mathcal{F}_a(y),$$

where

$$\mathcal{E}_a(y) := |y|^{\alpha-n} w(|y|) \sum_{k=0}^{\infty} \Phi(2^{-k}|y|) |a(y) - a_{B(0,r)}|$$

and

$$\mathcal{F}_a(y) := |y|^{\alpha-n} w(|y|) \sum_{k=0}^{\infty} \Phi(2^{-k}|y|) |a_{B(0,r)} - a_{B_k(|y|)}|.$$

For  $\mathcal{E}_a$  we have

$$\|\mathcal{E}_a\chi_{B(0,r)}\|_{L^q} \leq \sum_{k=0}^{\infty} \left( \int_{B(0,r)} |y|^{\alpha-n} w(|y|) \Phi(2^{-k}|y|) |a(y) - a_{B(0,r)}|^q dy \right)^{\frac{1}{q}}.$$

Applying the Hölder inequality with the exponent  $s > 1$ , we get

$$\begin{aligned} \|\mathcal{E}_a\chi_{B(0,r)}\|_{L^q} &\leq \sum_{k=0}^{\infty} \left( \int_{B(0,r)} \left| \frac{w(|y|)\Phi(2^{-k}|y|)}{|y|^{n-\alpha}} \right|^{qs'} dy \right)^{\frac{1}{qs'}} \left( \int_{B(0,r)} |a(y) - a_{B(0,r)}|^{qs} dy \right)^{\frac{1}{qs}} \\ &\leq C \|a\|_{qs}^{*,0} r^{\frac{n}{qs}} \sum_{k=0}^{\infty} \left( \int_{B(0,r)} \left| \frac{w(|y|)\Phi(2^{-k}|y|)}{|y|^{n-\alpha}} \right|^{qs'} dy \right)^{\frac{1}{qs'}} \\ &= C \|a\|_{qs}^{*,0} r^{\frac{n}{qs}} \sum_{k=0}^{\infty} \left( \int_0^r \left| \frac{w(t)\Phi(2^{-k}t)}{t^{n-\alpha}} \right|^{qs'} t^{n-1} dt \right)^{\frac{1}{qs'}} \end{aligned}$$

By (3.7) and (2.15), (2.16) we see that  $\frac{w(t)}{t^\beta}$  is almost decreasing and then we obtain

$$w(t) \leq c 2^{\beta k} w(2^{-k}t) \quad (3.9)$$

and consequently

$$\begin{aligned} \|\mathcal{E}_a\chi_{B(0,r)}\|_{L^q} &\leq C \|a\|_{qs}^{*,0} r^{\frac{n}{qs}} \sum_{k=0}^{\infty} 2^{k\beta} \left( \int_0^r \left| \frac{w(2^{-k}\tau)\Phi(2^{-k}\tau)}{\tau^{n-\alpha}} \right|^{qs'} \tau^{n-1} d\tau \right)^{\frac{1}{qs'}} \\ &= C \|a\|_{qs}^{*,0} r^{\frac{n}{qs}} \sum_{k=0}^{\infty} 2^{k\left(\beta+\alpha-\frac{n}{(qs')^\gamma}\right)} \left( \int_0^{2^{-k}r} \left| \frac{w(\tau)\Phi(\tau)}{\tau^{n-\alpha}} \right|^{qs'} \tau^{n-1} d\tau \right)^{\frac{1}{qs'}} \end{aligned}$$

$$\begin{aligned}
&= C \|a\|_{qs}^{*,0} r^{\alpha+\beta-\frac{n}{q'}} \sum_{k=0}^{\infty} \left(2^{-k} r\right)^{\frac{n}{(qs')^r}-\alpha-\beta} \\
&\quad \times \left( \int_0^{2^{-k} r} \tau^{(\alpha-\frac{n}{p})_{qs'}+n-1} \varphi_{\frac{q}{p}s'}^{\frac{q}{p}s'}(\tau) d\tau \right)^{\frac{1}{qs'}}.
\end{aligned}$$

Now we apply Lemma 2.4 with

$$g(t) = t^{\frac{n}{(qs')^r}-\alpha-\beta} \left( \int_0^t [t^{(\alpha-\frac{n}{p})_q} \varphi_{\frac{q}{p}}^{\frac{q}{p}}(t)]^{s'} t^{n-1} dt \right)^{\frac{1}{qs'}}. \quad (3.10)$$

To justify the application of Lemma 2.4, note that from the assumption of the theorem it follows that  $\varphi_{\frac{1}{p}} = \frac{\varphi_{\frac{p}{p}}}{w} w$  is a product of two functions in  $\Delta_2$ . Then  $\varphi_{\frac{1}{p}}$  and consequently the integrand in (3.10) is doubling, and then use Lemma 2.3. Therefore,

$$\|\mathcal{E}_a \chi_{B(0,r)}\|_{L^q} \leq C \|a\|_{ps}^{*,0} r^{\alpha+\beta-\frac{n}{q'}} \int_0^r \varrho^{\frac{n}{(qs')^r}-\alpha-\beta} \left( \int_0^{\varrho} [t^{\alpha-\frac{n}{p}} \varphi_{\frac{1}{p}}^{\frac{1}{p}}(t)]^{qs'} t^{n-1} dt \right)^{\frac{1}{qs'}} \frac{d\varrho}{\varrho},$$

which generates the second term in (3.6) with  $\mathbb{B}_{s,\beta}(r)$ .

For the term  $\mathcal{F}_a$ , by Lemma 2.1 we obtain

$$|\mathcal{F}_a(y)| \leq C \|a\|^{*,0} |y|^{\alpha-n} w(|y|) \sum_{k=0}^{\infty} \Phi(2^{-k}|y|) \left[ 1 + \ln \left( 2^k \frac{r}{|y|} \right) \right]$$

and consequently

$$\|\mathcal{F}_a \chi_{B(0,r)}\|_{L^q} \leq C \|a\|^{*,0} \sum_{k=0}^{\infty} \left( \int_{|y|<r} \left| |y|^{\alpha-n} w(|y|) \Phi(2^{-k}|y|) \left[ 1 + \ln \left( 2^k \frac{r}{|y|} \right) \right] \right|^q dy \right)^{\frac{1}{q}}.$$

Hence by (3.9) we have

$$\begin{aligned}
&\|\mathcal{F}_a \chi_B(0, r)\|_{L^q} \\
&\leq C \|a\|^{*,0} \sum_{k=0}^{\infty} 2^{\beta k} \left( \int_{|y|<r} \left| |y|^{\alpha-n} w(2^{-k}|y|) \Phi(2^{-k}|y|) \left[ 1 + \ln \left( 2^k \frac{r}{|y|} \right) \right] \right|^q dy \right)^{\frac{1}{q}} \\
&= C \|a\|^{*,0} \sum_{k=0}^{\infty} 2^{k(\alpha+\beta-n)} \left( \int_{|y|<r} \left| (2^{-k}|y|)^{\alpha-n} w(2^{-k}|y|) \Phi(2^{-k}|y|) \left[ 1 \right. \right. \right. \\
&\quad \left. \left. \left. + \ln \left( \frac{r}{2^{-k}|y|} \right) \right] \right|^q dy \right)^{\frac{1}{q}}
\end{aligned}$$

Passing to polar coordinates, we have

$$\begin{aligned}
 \|\mathcal{F}_a \chi_{B(0,r)}\|_{L^q} &= C \|a\|^{*,0} \sum_{k=0}^{\infty} 2^{k(\alpha+\beta-n)} \left( \int_0^r \left| (2^{-k}t)^{\alpha-n} w(2^{-k}t) \Phi(2^{-k}t) \left[ 1 \right. \right. \right. \\
 &\quad \left. \left. \left. + \ln \left( \frac{r}{2^{-k}t} \right) \right] \right|^q t^{n-1} dt \right)^{\frac{1}{q}} \\
 &= C \|a\|^{*,0} r^{\alpha+\beta-\frac{n}{q'}} \sum_{k=0}^{\infty} (2^{-k}r)^{\frac{n}{q'}-\alpha-\beta} \\
 &\quad \times \left( \int_0^{2^{-k}r} \left| \tau^{\alpha-n} w(\tau) \Phi(\tau) \ln \frac{er}{\tau} \right|^q \tau^{n-1} d\tau \right)^{\frac{1}{q}}.
 \end{aligned}$$

Now we apply Lemma 2.4 with

$$g(t) := t^{\frac{n}{q'}-\alpha-\beta} \left( \int_0^t \left| \tau^{\alpha-\frac{n}{p}} \varphi^{\frac{1}{p}}(\tau) \ln \frac{er}{\tau} \right|^q \tau^{n-1} d\tau \right)^{\frac{1}{q}}.$$

The application of Lemma 2.4 is justified by means of Lemma 2.3 similarly to arguments after the formula (3.10). Note that the dependence of  $g(t)$  in this case on the parameter  $r$  is not of importance since  $g$  is the product of a power function and an increasing function, so that the constant  $C$  in the estimate of Lemma 2.4 does not depend on  $r$ . We get

$$\|\mathcal{F}_a \chi_{B(0,r)}\|_{L^q} \leq C \|a\|^{*,0} r^{\alpha+\beta-\frac{n}{q'}} \int_0^r t^{\frac{n}{q'}-\alpha-\beta} \left( \int_0^t \left| \tau^{\alpha-\frac{n}{p}} \varphi^{\frac{1}{p}}(\tau) \ln \frac{er}{\tau} \right|^q \tau^{n-1} d\tau \right)^{\frac{1}{q}} \frac{dt}{t},$$

which generates the third term in (3.6) with  $\mathbb{C}_\beta(r)$  and completes the proof.  $\square$

In the sequel we use the notation

$$P_s := \max \{p', qs\}, \quad s > 1.$$

From Theorem 3.3 we immediately obtain the following boundedness result for commutators of the Hardy operator  $H_w^\alpha$ :

**Theorem 3.4.** *Let  $p, q, w, \varphi$  and  $a$  satisfy the assumptions of Theorem 3.3 and  $\psi$  satisfy the assumptions in (1.6). Then*

$$\|[a, H_w^\alpha]f\|_{\mathcal{L}^{q,\psi;0}} \leq C \|a\|_{P_s}^{*,0} \|f\|_{\mathcal{L}^{p,\varphi;0}},$$

for  $s > 1$ , if

$$\sup_{r>0} \frac{\mathbb{A}(r)}{\psi^{\frac{1}{q}}(r)} < \infty, \quad \sup_{r>0} \frac{\mathbb{B}_{s,\beta}(r)}{\psi^{\frac{1}{q}}(r)} < \infty \quad \text{and} \quad \sup_{r>0} \frac{\mathbb{C}_\beta(r)}{\psi^{\frac{1}{q}}(r)} < \infty. \quad (3.11)$$

The boundedness conditions (3.11) involve the integrals  $\mathbb{A}(r)$ ,  $\mathbb{B}_{s,\beta}(r)$  and  $\mathbb{C}_\beta(r)$  defined in (3.3)–(3.5). The integrals in (3.3)–(3.5) have the structure similar to the construction used in the definition of the class  $\Phi^\gamma$ . Consequently, the expressions for  $\mathbb{A}(r)$ ,  $\mathbb{B}_{s,\beta}(r)$  and  $\mathbb{C}_\beta(r)$  may be essentially simplified if we suppose that the corresponding functions under the integral sign are in the class  $\Phi^\gamma$  with an appropriate  $\gamma$ .

In the lemma below we provide such a simplification under some assumptions in  $\Phi^\gamma$ -terms. In this relation note that the functions integrated in (3.3)–(3.5) belong to some  $\Phi^\gamma(\mathbb{R}_+)$ , see (2.19). However, to simplify the expressions for  $\mathbb{A}(r)$ ,  $\mathbb{B}_{s,\beta}(r)$  and  $\mathbb{C}_\beta(r)$ , in (3.3)–(3.5) we need the classes  $\Phi^\gamma(\mathbb{R}_+)$  with concrete values of  $\gamma$ . Note that the appearing conditions in  $\Phi^\gamma$ -terms are not much restrictive in comparison with (3.11), because they are very close in fact to the requirement that the integrals in (3.3)–(3.5) converge.

First we need the following lemma:

**Lemma 3.5.** *Let  $v \in \overline{W}(\mathbb{R}_+)$ ,  $m(v) > 0$  and  $q \in \mathbb{R}$ . Then*

$$\int_0^t v(s) \left(1 + \ln \frac{r}{s}\right)^q \frac{ds}{s} \leq C \left(1 + \ln \frac{r}{t}\right)^q v(t), \quad (3.12)$$

where  $0 < t \leq r < \infty$  and  $C$  does not depend on  $r$  and  $t$ .

*Proof.* Let  $\nu$  be any number such that  $0 < \nu < m(v)$ . We have

$$\int_0^t v(s) \left(1 + \ln \frac{r}{s}\right)^q \frac{ds}{s} = \int_0^t \frac{v(s)}{s^\nu} s^\nu \left(1 + \ln \frac{r}{s}\right)^q \frac{ds}{s}.$$

Since  $\frac{v(s)}{s^\nu}$  is almost increasing, we have that

$$\int_0^t v(s) \left(1 + \ln \frac{r}{s}\right)^q \frac{ds}{s} \leq C \frac{v(t)}{t^\nu} \int_0^t s^\nu \left(1 + \ln \frac{r}{s}\right)^q \frac{ds}{s}.$$

After the change of variables  $\frac{r}{s} = \frac{1}{\xi}$

$$\int_0^t v(s) \left(1 + \ln \frac{r}{s}\right)^q \frac{ds}{s} \leq C \frac{v(t)}{t^\nu} r^\nu \int_0^{\frac{t}{r}} \xi^\nu \left(1 + \ln \frac{1}{\xi}\right)^q \frac{d\xi}{\xi}.$$

For the function  $g(\xi) = \xi^\nu \left(1 + \ln \frac{1}{\xi}\right)^q$  we see that  $m(g) = \nu$ ; then

$$\int_0^{\frac{t}{r}} g(\xi) \frac{d\xi}{\xi} \leq C g\left(\frac{t}{r}\right) = C \left(\frac{t}{r}\right)^\nu \left(1 + \ln \frac{r}{t}\right)^q,$$

which proves the lemma.  $\square$

**Theorem 3.6.** *Let  $1 < p < \infty$ ,  $1 < q < \infty$ ,  $a \in CMO_{p'} \cap CMO_{sq}$  for some  $s > 1$ .  $\varphi$  and  $\psi$  satisfy the assumptions in (1.6) and*

$$w \in \underline{W}(\mathbb{R}_+), \quad \varphi \in \Phi^\gamma, \quad \gamma = np \left( \frac{1}{p} - \frac{1}{q} - \frac{\alpha}{n} \right) \quad \text{and} \quad \frac{\varphi^{1/p}}{w} \in \Delta_2. \quad (3.13)$$

Then

$$\| [a, H_w^\alpha] f \|_{\mathcal{L}^{q, \psi; 0}} \leq C \| a \|_{P_s^{*, 0}}^* \| f \|_{\mathcal{L}^{p, \varphi; 0}},$$

if

$$\sup_{r>0} r^{\alpha-\frac{n}{p}+\frac{n}{q}} \frac{\varphi^{\frac{1}{p}}(r)}{\psi^{\frac{1}{q}}(r)} < \infty \quad (3.14)$$

and

$$M(w) < \frac{n}{p'} + \frac{m(\varphi)}{p}. \quad (3.15)$$

*Proof.* We have to apply Theorem 3.4 via verification of the conditions (3.11).

*Step 1°: Estimation of  $\mathbb{A}(r)$ .* It is easily seen that the condition (3.15) guarantees the inclusion  $\frac{\varphi^{1/p}}{w} \in \Phi^{-\frac{n}{p'}}$  in view of the property  $m\left(\frac{u}{v}\right) \geq m(u) - M(v)$  of the indices. Therefore, the inner integral in (3.3) is dominated by  $C t^{\frac{n}{p'}-\frac{1}{p}} \frac{\varphi^{\frac{1}{p}}}{w}$ . Therefore,

$$\mathbb{A}(r) \leq C \left( \int_0^r t^{(\alpha-\frac{n}{p})q+n} \varphi^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}}$$

Taking into account the property (2.18) and the fact that  $\Phi^{np(\frac{1}{p}-\frac{1}{qs'}-\frac{\alpha}{n})} \subset \Phi^{np(\frac{1}{p}-\frac{1}{q}-\frac{\alpha}{n})}$ , we have

$$\mathbb{A}(r) \leq Cr^{\alpha-\frac{n}{p}+\frac{n}{q}} \varphi^{\frac{1}{p}}.$$

*Step 2°: Estimation of  $\mathbb{B}_{s,\beta}(r)$ .* The estimation is similar to that at Step 1°. First we get rid of the inner integral in (3.4) thanks to the  $\Phi^\gamma$  assumption on  $\varphi$  and get

$$\mathbb{B}_{s,\beta}(r) \leq Cr^{\alpha+\beta-\frac{n}{q'}} \int_0^r t^{\frac{n}{p'}-\beta} \varphi^{\frac{1}{p}} \frac{dt}{t}.$$

Now we wish to use the fact that  $\varphi^{\frac{1}{p}} \in \Phi^{\beta-\frac{n}{p'}}$ . This may be possible under the appropriate choice of  $\beta$ . Indeed, by the property (2.17) we should have the inequality  $\frac{1}{p} \min\{m(\varphi), m_\infty(\varphi)\} > \beta - \frac{n}{p'}$ , i.e.  $\beta < \frac{n}{p'} + \frac{\min\{m(\varphi), m_\infty(\varphi)\}}{p}$ . Up to now we had the only restriction  $\beta > M(w)$  imposed on  $\beta$  in Theorem 3.3. Consequently, the required choice of  $\beta$  is possible in view of the assumption (3.15), and then we obtain

$$\mathbb{B}_{s,\beta}(r) \leq Cr^{\alpha-\frac{n}{p}+\frac{n}{q}} \varphi^{\frac{1}{p}}.$$

*Step 3°: Estimation of  $\mathbb{C}(r)$ .* This time we need Lemma 3.5. Applying this lemma with  $v(\tau) = \tau^{(\alpha-\frac{n}{p})q+n} \varphi^{\frac{q}{p}}(\tau)$  in the inner integral in (3.5) we get

$$\mathbb{C}_\beta(r) \leq Cr^{\alpha+\beta-\frac{n}{q'}} \int_0^r t^{\frac{n}{p'}-\beta} \varphi^{\frac{1}{p}}(t) \left(1 + \ln \frac{r}{t}\right) \frac{dt}{t}.$$

The condition  $m(v) > 0$  of Lemma 3.5 is satisfied by the  $\Phi^\gamma$  assumption on  $\varphi$  in our theorem. It remains to apply Lemma 3.5 again, with the function  $v(t) = t^{\frac{n}{p'}-\beta} \varphi^{\frac{1}{p}}(t)$ . The condition  $m(v) > 0$  is guaranteed by the choice

$\beta < \frac{n}{p'} + \frac{\min\{m(\varphi), m_\infty(\varphi)\}}{p}$  as above in the estimation of  $\mathbb{B}_{s,\beta}(r)$ . Consequently, we get

$$\mathbb{C}(r) \leq Cr^{\alpha - \frac{n}{p} + \frac{n}{q}} \varphi^{\frac{1}{p}}.$$

*Step 4°:* Gathering the obtained estimates for  $\mathbb{A}(r)$ ,  $\mathbb{B}_{s,\beta}(r)$  and  $\mathbb{C}(r)$ , from (3.6) we obtain

$$\| [a, H_w^\alpha] f \|_{L^q(B(0,r))} \leq C \| a \|_{P_s^{*,0}}^* r^{\alpha - \frac{n}{p} + \frac{n}{q}} \varphi^{\frac{1}{p}} \| f \|_{\mathcal{L}^{p,\varphi;0}},$$

whence the statement of the theorem follows.  $\square$

*Remark 3.7.* Assumptions on  $\varphi$  and  $\frac{\varphi^{\frac{1}{p}}}{w}$  in (3.13) on inclusion of these functions into the corresponding  $\Phi^\gamma$  classes may be equivalently written in terms of the numerical inequalities of their indices according to the property (2.17).

*Remark 3.8.* In condition (3.14) we may in fact chose

$$r^{\alpha - \frac{n}{p} + \frac{n}{q}} \frac{\varphi^{\frac{1}{p}}(r)}{\psi^{\frac{1}{q}}(r)} \equiv 1, \quad (3.16)$$

since any other choice of  $\psi$  in (3.14) may be only a worsening of (3.16). The choice (3.16), i.e.

$$\psi^{\frac{1}{q}}(r) = r^{\alpha - \frac{n}{p} + \frac{n}{q}} \varphi^{\frac{1}{p}}(r)$$

defines the function  $\psi$  for given  $p, q$  and  $\varphi$ . In particular, when  $\alpha p < n$  and we chose  $q$  as the Sobolev exponent, i.e.  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ , we have  $\psi = \varphi^{\frac{1}{p}}$ . The possibility to have the coinciding functions  $\psi(r) \equiv \varphi(r)$  arises only in the case of the power function  $\varphi(r) = r^\lambda$ , i.e. in the case of classical Morrey spaces and corresponds to Adams' exponent:  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$  for  $\alpha p < n - \lambda$ . This situation is reflected in the corollary below.

For the case of the classical Morrey spaces, i.e.  $\varphi(r) = r^\lambda$  and the Sobolev–Adams exponent, i.e.  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$ , from Theorem 3.6 we derive the following corollary:

**Corollary 3.9.** *Let  $0 < \lambda < n$ ,  $1 < p < \frac{n-\lambda}{\alpha}$ ,  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$ ,  $\frac{1}{w} \in \Delta_2 \cap \overline{W}(\mathbb{R}_+)$  and  $M(w) < \frac{n}{p'} + \frac{\lambda}{p}$ . Let also  $a \in CMO_{P_s}$  with  $s > 1$ . Then*

$$\| [a, H_w^\alpha] f \|_{\mathcal{L}^{q,\lambda;0}} \leq C \| a \|_{P_s^{*,0}}^* \| f \|_{\mathcal{L}^{p,\lambda;0}}, \quad (3.17)$$

In particular, in Corollary 3.9 one can take a weight  $w(t) = t^\gamma(1+|nt|)^\mu$  with  $\nu < \frac{n}{p'}$  and  $\mu \in \mathbb{R}$ .

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